ON A GENERALIZATION OF A RESULT OF PESKINE AND SZPIRO

TONY J. PUTHENPURAKAL

ABSTRACT. Let $(R, \mathfrak{m})$ be a regular local ring containing a field K. Let I be a Cohen-Macaulay ideal of height g. If $\operatorname{char} K = p > 0$ then by a result of Peskine and Szpiro the local cohomology modules $H^i_I(R)$ vanish for $i > g$. This result is not true if $\operatorname{char} K = 0$. However we prove that the Bass numbers of the local cohomology module $H^g_I(R)$ completely determine whether $H^i_I(R)$ vanish for $i > g$.

The result of this paper has been proved more generally for Gorenstein local rings by Hellus and Schenzel [3, Theorem 3.2]. *However our result for regular rings is elementary to prove. In particular we do not use spectral sequences in our proof.*

1. INTRODUCTION

Let $(R, \mathfrak{m})$ be a regular local ring containing a field K. Motivated by a result of Peskine and Szpiro we make the following:

Definition 1.1. An ideal I of R is said to be a *Peskine-Szpiro ideal* of R if

(1) I is a Cohen-Macaulay ideal.
(2) $H^i_I(R) = 0$ for all $i \neq \operatorname{height} I$.

Note that as $\operatorname{height} I = \operatorname{grade} I$ we have $H^i_I(R) = 0$ for $i < \operatorname{height} I$. Thus the only real condition for a Cohen-Macaulay ideal I to be a Peskine-Szpiro ideal is that $H^i_I(R) = 0$ for $i > \operatorname{height} I$. In their fundamental paper [7, Proposition III.4.1] Peskine and Szpiro proved that if $\operatorname{char} K = p > 0$ then for all Cohen-Macaulay ideals I the local cohomology modules $H^i_I(R)$ vanish for $i > \operatorname{height} I$. This result is not true if $\operatorname{char} K = 0$, for instance see [2, Example 21.31]. We prove the following surprising result:

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d containing a field K. Let I be a Cohen-Macaulay ideal of height g. The following conditions are equivalent:*

(i) *I is a Peskine-Szpiro ideal of R.*
(ii) *For any prime ideal P of R containing I, the Bass number*

$$\mu_i(P, H^g_I(R)) = \begin{cases} 1 & \text{if } \ i = \operatorname{height} P - g, \\ 0 & \text{otherwise.} \end{cases}$$

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Here the j^{th} Bass number of an R-module M with respect to a prime ideal P is defined as $\mu_j(P, M) = \dim_{k(P)} \operatorname{Ext}^j_{R_P}(k(P), M_P)$ where $k(P)$ is the residue field of R_P. Our result is essentially only an observation.

1.3. We need the following remarkable properties of local cohomology modules over regular local rings containing a field (proved by Huneke and Sharp [4] if $\operatorname{char} K = p > 0$ and by Lyubeznik [5] if $\operatorname{char} K = 0$). Let $(R, \mathfrak{m})$ be a regular ring containing a field K and I is an ideal in R. Then the local cohomology modules of R with respect to I have the following properties:

(i) $H^j_{\mathfrak{m}}(H^i_I(R))$ is injective.
(ii) $\operatorname{injdim}_R H^i_I(R) \leq \dim \operatorname{Supp} H^i_I(R)$.
(iii) The set of associated primes of $H^i_I(R)$ is finite.
(iv) All the Bass numbers of $H^i_I(R)$ are finite.

Here $\operatorname{injdim}_R H^i_I(R)$ denotes the injective dimension of $H^i_I(R)$. Also $\operatorname{Supp} M = \{P \mid M_P \neq 0 \text{ and } P \text{ is a prime in } R\}$ is the support of an R-module M.

2. Permanence properties of Peskine-Szpiro ideals

In this section we prove some permanence properties of Peskine-Szpiro ideals. We also show that if $\dim R - \operatorname{height} I \leq 2$ then a Cohen-Macaulay ideal I is a Peskine-Szpiro ideal.

Proposition 2.1. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field K. Let I be a Peskine-Szpiro ideal of R. Let $g = \operatorname{height} I$.*

(1) *Let P be a prime ideal in R containing I. Then I_P is a Peskine-Szpiro ideal of R_P.*
(2) *Assume $g < \dim R$. Let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ be R/I-regular. Then the ideal $(I + (x))/(x)$ is a Peskine-Szpiro ideal of $R/(x)$.*

Proof. (1) Note I_P is a Cohen-Macaulay ideal of height g in R_P. Also note that for $i \neq g$ we have

$$H^i_{I_P}(R_P) = H^i_I(R)_P = 0.$$

Thus I_P is a Peskine-Szpiro ideal of R_P.

(2) Note that $J = (I + (x))/(x)$ is a Cohen-Macaulay ideal of height g in the regular ring $\overline{R} = R/(x)$. The short exact sequence

$$0 \to R \xrightarrow{x} R \to \overline{R} \to 0,$$

induces a long exact sequence

$$\cdots \to H^i_I(R) \to H^i_J(\overline{R}) \to H^{i+1}_I(R) \to \cdots .$$

Thus $H^i_J(\overline{R}) = 0$ for $i > g$. Therefore J is a Peskine-Szpiro ideal of $\overline{R}$. $\square$

We now show that Cohen-Macaulay ideals of small dimensions are Peskine-Szpiro. This result is already known, However we give a proof due to lack of a suitable reference.

Proposition 2.2. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field K. Let I be a Cohen-Macaulay ideal with $\dim R - \operatorname{height} I \leq 2$. Then I is a Peskine-Szpiro ideal.*

Proof. We have nothing to show if $\operatorname{char} K = p > 0$. So we assume $\operatorname{char} K = 0$. Let $\dim R = d$ and $\operatorname{height} I = g$.

If $g = d$ then I is $\mathfrak{m}$-primary. By Grothendieck vanishing theorem we have $H^i_I(R) = 0$ for $i > d$. So I is a Peskine-Szpiro ideal.

Now consider the case when $g = d-1$. Note $\dim R/I = 1$. So $\dim \widehat{R}/I\widehat{R} = 1$. By Hartshorne-Lichtenbaum theorem, cf. [2, Theorem 14.1], we have that $H^d_{I\widehat{R}}(\widehat{R}) = 0$. By faithful flatness we get $H^d_I(R) = 0$.

Finally we consider the case when $g = d-2$. We choose a flat extension $(B, \mathfrak{n})$ of R with $\mathfrak{m}B = \mathfrak{n}$, B complete and $B/\mathfrak{n}$ algebraically closed. We note that B/IB is Cohen-Macaulay and $\dim B/IB = 2$. As B/IB is Cohen-Macaulay we get that the punctured spectrum $\operatorname{Spec}^\circ(B/IB)$ is connected see [2, Proposition 15.7]. So $H^{d-1}_{IB}(B) = 0$ by a result due to Ogus [6, 2.11]. By faithful faithfulness we get $H^{d-1}_I(R) = 0$. By an argument similar to the previous case we also get $H^d_I(R) = 0$. $\square$

3. Proof of Theorem 1.2

In this section we prove our main result. The following remarks are relevant.

Remark 3.1. (1) Notice for any ideal J of height g we have $\operatorname{Ass} H^g_J(R) = \{P \mid P \supset J \text{ and } \operatorname{height} P = g\}$. Also for any such prime ideal P we have $\mu_0(P, H^g_J(R)) = 1$.

(2) Let I be a Cohen-Macaulay ideal of height g in a regular local ring. Let P be an ideal of height $g+r$ and containing I. We note that $\dim H^g_I(R)_P = r$. So by 1.3 we get $\operatorname{injdim}_{R_P} H^g_I(R)_P \leq r$. Thus $\mu_i(P, H^g_I(R)) = 0$ for $i > r$.

Let us recall the following result due to Rees, cf. [1, 3.1.16].

3.2. Let S be a commutative ring and let M and N be S-modules. (We note that S need not be Noetherian. Also M, N need not be finitely generated as S-modules.) Assume there exists $x \in S$ such that it is $S \oplus M$-regular and $xN = 0$. Set $T = S/(x)$. Then $\operatorname{Hom}_S(N, M) = 0$ and for $i \geq 1$ we have

$$\operatorname{Ext}^i_S(N, M) \cong \operatorname{Ext}^{i-1}_T(N, M/xM).$$

We now give:

Proof of Theorem 1.2. We first prove (i) $\implies$ (ii). So I is a Peskine-Szpiro ideal. We prove our result by induction on $d - g$.

If $d - g = 0$ then I is $\mathfrak{m}$-primary. So $H^d_I(R) = H^d_\mathfrak{m}(R) = E_R(R/\mathfrak{m})$ the injective hull of the residue field. Clearly $\mu_0(\mathfrak{m}, H^d_I(R)) = 1$ and $\mu_i(\mathfrak{m}, H^d_I(R)) = 0$ for $i \geq 1$.

Now assume $d-g = 1$. If P is a prime ideal of R containing I with $\operatorname{height} P = d-1$ then by 3.1 we have $\mu_0(P, H^{d-1}_I(R)) = 1$ and $\mu_i(P, H^{d-1}_I(R)) = 0$ for $i \geq 1$. We now consider the case when $P = \mathfrak{m}$. By 3.1 we have $\mu_0(\mathfrak{m}, H^{d-1}_I(R)) = 0$. Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ which is R/I-regular. Set $\overline{R} = R/(x)$, $\mathfrak{n} = \mathfrak{m}/(x)$ and $J = I\overline{R} = (I + (x))/(x)$. Then J is $\mathfrak{n}$-primary. The exact sequence $0 \to R \xrightarrow{x} R \to \overline{R} \to 0$ induces the following exact sequence in cohomology

$$0 \to H^{d-1}_I(R) \xrightarrow{x} H^{d-1}_I(R) \to H^{d-1}_J(\overline{R}) \to 0.$$

Here we have used that I is a Peskine-Szpiro ideal and J is $\mathfrak{n}$-primary. Thus by 3.2 we have for $i \geq 1$,

$$\mu_i(\mathfrak{m}, H^{d-1}_I(R)) = \mu_{i-1}(\mathfrak{n}, H^{d-1}_J(\overline{R})) = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the result follows in this case.

Now consider the case when $d - g \geq 2$. Let P be a prime ideal in R containing I of height $g + r$. We first consider the case when $P \neq \mathfrak{m}$. By 2.1 we get that I_P is a Peskine-Szpiro ideal of height g in R_P. Also $\dim R_P - g < d - g$. So by induction hypothesis we have

$$\mu_i(P, H^i_I(R)) = \mu_i(PR_P, H^i_{IR_P}(R_P)) = \begin{cases} 1 & \text{if } i = r, \\ 0 & \text{otherwise.} \end{cases}$$

□

We now consider the case when $P = \mathfrak{m}$. By 3.1 we have $\mu_0(\mathfrak{m}, H^g_I(R)) = 0$. Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ which is R/I-regular. Set $\overline{R} = R/(x)$, $\mathfrak{n} = \mathfrak{m}/(x)$ and $J = I\overline{R} = (I + (x))/(x)$. Then J is height g Peskine-Szpiro ideal in $\overline{R}$, see 2.1. The exact sequence $0 \to R \xrightarrow{x} R \to \overline{R} \to 0$ induces the following exact sequence in cohomology

$$0 \to H^g_I(R) \xrightarrow{x} H^g_I(R) \to H^g_J(\overline{R}) \to 0.$$

Here we have used that I is a Peskine-Szpiro ideal in R and J is Peskine-Szpiro ideal in $\overline{R}$. Thus by 3.2 we have for $i \geq 1$,

$$\mu_i(\mathfrak{m}, H^g_I(R)) = \mu_{i-1}(\mathfrak{n}, H^g_J(\overline{R})) = \begin{cases} 1 & \text{if } i - 1 = d - 1 - g, \\ 0 & \text{otherwise.} \end{cases}$$

For the latter equality we have used induction hypothesis on the Peskine-Szpiro ideal J (as $\dim \overline{R} - \operatorname{height} J = d - 1 - g$). We note that $i - 1 = d - 1 - g$ is same as $i = d - g$. Thus we have

$$\mu_i(\mathfrak{m}, H^g_I(R)) = \begin{cases} 1 & \text{if } i = d - g, \\ 0 & \text{otherwise.} \end{cases}$$

We now prove (ii) $\implies$ (i). By Peskine and Szpiro's result we may assume $\operatorname{char} K = 0$. We prove the result by induction on $d - g$. If $d - g \leq 2$ then the result holds by Proposition 2.2. So we may assume $d - g \geq 3$. Let P be a prime ideal in R containing I with $P \neq \mathfrak{m}$. The ideal I_P is a Cohen-Macaulay ideal of height g in R_P satisfying the condition (ii) on Bass numbers of $H^g_{I_P}(R_P)$. As $\dim R_P - g < d - g$ we get by our induction hypothesis that I_P is a Peskine-Szpiro ideal in R_P. Thus $H^i_{I_P}(R_P) = 0$ for $i > g$. It follows that $\operatorname{Supp} H^i_I(R) \subseteq \{\mathfrak{m}\}$ for $i > g$. Let $k = R/\mathfrak{m}$ and let $E_R(k)$ be the injective hull of k as a R-module. Then by 1.3 there exists non-negative integers r_i with

$$H^i_I(R) = E_R(k)^{r_i} \quad \text{for } i > g. \tag{3.2.1}$$

Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ which is R/I-regular. Set $\overline{R} = R/(x)$, $\mathfrak{n} = \mathfrak{m}/(x)$ and $J = I\overline{R} = (I + (x))/(x)$. Then J is height g Cohen-Macaulay ideal in $\overline{R}$. The exact sequence $0 \to R \xrightarrow{x} R \to \overline{R} \to 0$ induces the following exact sequence in cohomology

$$0 \to H^g_I(R) \xrightarrow{x} H^g_I(R) \to H^g_J(\overline{R}) \to H^{g+1}_I(R) \xrightarrow{x} H^{g+1}_I(R) \to \cdots \tag{3.2.2}$$

We consider two cases:
Case 1 : $H_I^{g+1}(R) \neq 0$.
We note that $\operatorname{Hom}_R(\overline{R}, E_R(k)) = E_{\overline{R}}(k)$. Thus the short exact sequence $0 \to R \xrightarrow{x} R \to \overline{R} \to 0$ induces an exact sequence

$$0 \to E_{\overline{R}}(k) \to E_R(k) \xrightarrow{x} E_R(k) \to 0. \tag{3.2.3}$$

By (3.2.1) and (3.2.3) the exact sequence (3.2.2) breaks down into two exact sequences

$$0 \to H_I^g(R) \xrightarrow{x} H_I^g(R) \to V \to 0, \tag{3.2.4}$$

$$0 \to V \to H_J^g(\overline{R}) \to E_{\overline{R}}(k)^{r_{g+1}} \to 0. \tag{3.2.5}$$

As J is a Cohen-Macaulay ideal in $\overline{R}$ with $\dim \overline{R}/J = d-1-g \geq 2$ we get by 3.1 that $\mathfrak{n} \notin \operatorname{Ass}_{\overline{R}} H_J^g(\overline{R})$. It follows from (3.2.5) that $\mu_1(\mathfrak{n}, V) \geq r_{g+1} > 0$. By (3.2.4) and 3.2 we get that

$$\mu_2(\mathfrak{m}, H_I^g(R)) = \mu_1(\mathfrak{n}, V) > 0.$$

So by our hypothesis we get $d - g = 2$. This is a contradiction as we assumed $d - g \geq 3$.
Case 2 : $H_I^{g+1}(R) = 0$.
By (3.2.2) we get a short exact sequence,

$$0 \to H_I^g(R) \xrightarrow{x} H_I^g(R) \to H_J^g(\overline{R}) \to 0.$$

Again by 3.2 we get that the Cohen-Macaulay ideal J of $\overline{R}$ satisfies the conditions (ii) of our Theorem. As $\dim \overline{R} - \operatorname{height} J = d-g-1$ we get by induction hypothesis that J is Peskine-Szpiro ideal in $\overline{R}$. Thus $H_J^i(\overline{R}) = 0$ for $i > g$. Using (3.2.1) and (3.2.3) it follows that $H_I^i(R) = 0$ for $i \geq g+2$. Also by our assumption $H_I^{g+1}(R) = 0$. Thus I is a Peskine-Szpiro ideal of R.

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Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400 076, India
E-mail address: tputhen@math.iitb.ac.in